"Entropy in the universe is always increasing. At some point, the universe will reach its maximum state of entropy and then no work can be done. The universe will become a cold, lifeless place. This is known as the heat death of the universe. Get ready, it's coming..."

- Prof. Seamus C. Davis, U.C. Berkeley.

If you have any questions, suggestions or corrections to the solutions, don't hesitate to e-mail me at dfk@uclink4.berkeley.edu!

Problem 1 *Rohlf 12.5*

Consider a system of 6 spin-1/2 fermions having a total energy of 10 units. The fermions are in a quantum mechanical system where the ground state has 0 energy units, the first excited state has 1 energy unit, the second excited state has 2 energy units, etc. Determine the energy distribution function df/dE. Make an estimate of the Fermi energy.

We make a table of the possible distributions and the spin degeneracy of each state (i.e., if there is an isolated electron, it can be either spin up or spin down). We then total up number of times an electron is found in a state, and from this total we arrive at our distribution function f, which describes the probability to find an electron in a particular energy level. The total number of times we find an electron in an energy level for a distinct distribution (including spin degeneracy) is shown on the bottom line.

Degen.	**Energy**						
	0	**1**	**2**	**3**	**4**	**5**	**6**
4	2	2	1	0	0	0	1
4	2	2	0	1	0	1	0
1	2	2	0	0	2	0	0
16	2	1	1	1	1	0	0
4	2	1	2	0	0	1	0
1	2	0	2	2	0	0	0
4	1	2	1	2	0	0	0
4	1	2	2	0	1	0	0
TOTAL	68	54	42	30	22	8	4

f, which in this case is discrete, is just the ratio of the various total number of times a particle is found in an energy level in one of the distributions to the ground state ($E = 0$) number. So $f(E)$ is a discrete function described by the following relations:

$$f(0) = 1$$
$$f(1) = 0.79$$
$$f(2) = 0.62$$
$$f(3) = 0.44$$
$$f(4) = 0.32$$
$$f(5) = 0.12$$
$$f(6) = 0.6$$
$$f(E > 6) = 0$$

The Fermi energy E_F is where this distribution has $f(E_F) \approx 1/2$. This is seen from the Fermi-Dirac distribution:

$$f_{FD}(E) = \frac{1}{e^{(E-E_F)/(kT)} + 1}$$

$$f_{FD}(E_F) = 1/2.$$

This is around $E = 3$ for this system. It doesn't work out exactly because this is a discrete system with a small number of possible distributions. If the number of particles was greatly increased, the distribution would become increasingly well-described by f_{FD}, which is derived in the large N limit.

Problem 2

The energy of the 3D infinite square well in this case is given by:

$$E_n = \frac{\hbar^2\pi^2}{2mL^2}\left(n_x^2 + n_y^2 + \frac{n_z^2}{\epsilon^2}\right).$$

ϵ is small, so since the number of particles is chosen to be small ($\epsilon N \ll 1$), the temperature must be large in order to excite states with $n_z \neq 1$. We consider the case of low temperature, where always $n_z = 1$. In this case, the problem is effectively 2D, with the energies:

$$E_n = \frac{\hbar^2\pi^2}{2mL^2}\left(n_x^2 + n_y^2 + \frac{1}{\epsilon^2}\right).$$

We seek the difference Δ between the ground state energy and the Fermi energy, both of which have a term

$$\frac{\hbar^2\pi^2}{2mL^2\epsilon^2}$$

which will cancel out.

First, let's calculate the Fermi energy. The Fermi-Dirac distribution at $T = 0$ is given by

$$f_{FD} = 1 \quad E < E_F$$
$$f_{FD} = 0 \quad E > E_F.$$

The electrons will try to achieve the lowest energy possible, filling up states in accordance with the Pauli exclusion principle. The total number of particles is given by:

$$N = \int_0^\infty 2\frac{dN}{dE} f_{FD}(E)dE$$

where the 2 is for spin degeneracy of the electrons. At $T = 0$, this integral becomes:

$$N = \int_0^{E_F} 2\frac{dN}{dE}dE.$$

Now we must determine the density of states for 2D. The energy of a state (neglecting the common factor $\frac{\hbar^2\pi^2}{2mL^2\epsilon^2}$) is $\propto N^2$ where $N^2 = n_x^2 + n_y^2$ and is the square of the total number of available states. The derivative of N^2 with respect to energy is

$$\frac{d}{dE}N^2 = \frac{1}{4}2\pi N\frac{dN}{dE}$$

where the factor of $1/4$ arises because we consider only positive n_x, n_y and the 2π is from the integration about a ring of thickness dN in n-space. From this we can explicitly solve for $\frac{dN}{dE}$:

$$\frac{dN}{dE} = \frac{4mL^2}{\hbar^2\pi^3}\frac{1}{N} = \frac{2\sqrt{2m}L}{\hbar\pi^2}E^{-1/2}.$$

Now we employ this expression for $\frac{dN}{dE}$ in our integral for the total number of particles:

$$N = \int_0^{E_F} 2\frac{2\sqrt{2m}L}{\hbar\pi^2}E^{-1/2}dE = 8\frac{\sqrt{2m}L}{\hbar\pi^2}E_F^{1/2}.$$

Solving for E_F yields

$$\boxed{E_F = \frac{\hbar^2\pi^4N^2}{128mL^2}}$$

Problem 3

We start with the knowledge that the density of states is proportional to $E^{1/2}$ and the probability of occupation is

$$P(E) = \frac{1}{e^{\beta(E-E_F)} + 1}.$$

Let

$$\frac{dN}{dE} = cE^{1/2}.$$

where c is a constant. The fraction of nonrelativistic fermions in a gas of finite temperature T above the Fermi energy is given by the integral:

$$\mathcal{F} = \frac{\int_{E_F}^\infty P(E)\frac{dN}{dE}dE}{\int_0^\infty P(E)\frac{dN}{dE}dE}.$$

The denominator of the above equation is just the total number of particles, which is easiest to evaluate at $T = 0$ where, since $P(E)$ is just the Fermi-Dirac distribution function, we have:

$$P(E) = 1 \quad E < E_F$$
$$P(E) = 0 \quad E > E_F.$$

So the denominator is simply

$$\int_0^{E_F}\frac{dN}{dE}dE = \int_0^{E_F} cE^{1/2}dE = \frac{2}{3}cE_F^{3/2}.$$

So employing all of our information, the simplest expression we can get for $\mathcal{F}$ is:

$$\boxed{\mathcal{F} = \frac{3}{2}E_F^{-3/2}\int_{E_F}^\infty \frac{E^{1/2}dE}{e^{\beta(E-E_F)} + 1}}$$

Problem 4 *Rohlf 12.16*

We want to deduce the expression for density of states of a relativistic electron gas. We first get the density of states with respect to k (where $\hat{k}$ is the electron wave vector) in the usual manner for a 3D particle in a box problem. The volume of a shell of thickness dk in k-space (considering only positive k_x, k_y, and k_z) is $(4\pi k^2 dk)/8$. From the boundary conditions $k_i = (\pi/L)n_i$ (where $i = x, y, z$ and the box is length L on a side), we know the number of states per unit volume of k-space is $(L/\pi)^3$. So

$$\frac{dN}{dk} = \frac{k^2 V}{2\pi^2}$$

where V is the volume of the box. Dividing by volume, to get the density of states per unit volume, and converting to momentum p using the deBroglie relation $p = \hbar k$, we have:

$$\frac{dn}{dp} = \frac{4\pi p^2}{h^3}.$$

To convert this expression into density of states per unit energy $\rho(E)$, we use:

$$\rho(E) = \frac{dn}{dE} = \frac{dn}{dp}\frac{dp}{dE}.$$

The relativistic expression for momentum in terms of energy is:

$$p = \frac{1}{c}\sqrt{E^2 - m^2c^4}.$$

Therefore

$$\frac{dp}{dE} = \frac{E}{c^2 p}.$$

We now solve for $\rho(E)$:

$$\rho(E) = 2\frac{4\pi pE}{c^2h^3}$$

where the factor of 2 is for spin degeneracy.
The relativistic momentum is $p = \gamma m v$ and $v \approx c$. The relativistic energy is $E = \gamma mc^2$. Substituting these expressions in,

$$\boxed{\rho(E) = 2\left(\frac{4\pi m^2 c}{h^3}\right)\gamma^2}$$

Problem 5 *Rohlf 17.27*

(a)

There are six states of charmonium with $n = 2$. Charmonium is a bound state of a charm and anti-charm quark. We don't have to worry about symmetrization of the wavefunction because these are *not* identical fermions. The total spin s of the system can be:

$$s = s_1 + s_2 = 1, 0$$

where $s_1 = 1/2$ and $s_2 = 1/2$ are the spins of the quarks. The orbital angular momentum l of charmonium can be $l = 1, 0$ in the $n = 2$ state. The total (spin + orbital) angular momentum is given by

$$J = L + S = l + s,\ \ldots\ , |l - s|.$$

So we have the following possible states:

$$\begin{array}{ll} j = 2,1,0 & \text{for } s = 1, l = 1 \\ j = 1 & \text{for } s = 0, l = 1 \\ j = 1 & \text{for } s = 1, l = 0 \\ j = 0 & \text{for } s = 0, l = 0 \end{array}$$

which total six.

(b)

The unobserved state has $s = 0$, $l = 1$ and $j = 1$ by inspection.

(c)

By analogy to similar states in the chart of charmonium, the energy difference between states with aligned vs. anti-aligned spins of the quarks is ~ 100 MeV (the energy difference of the $\psi(2s)$ and $\eta_c(2s)$ states and $\psi(1s)$ and $\eta_c(1s)$ states, see Rohlf pg. 494). One could imagine that the energy splitting arises because of some spin-spin interaction between the quarks, so the $s = 0$, $l = 1$, $j = 1$ state of charmonium (called the ☺ or smiley particle) should be split in energy from the $\chi_{c1}(2p)$ state by ~ 100 MeV. So the mass of ☺ should be roughly 3400 MeV.

(d)

The ☺ particle is not observed because the method by which all the states of charmonium were observed involved creation of a $\psi(2s)$ particle and subsequent electromagnetic decay. The energy of emitted photons was measured and the spectrum of charmonium particles was established. Note that decays of $\psi(2s) \rightarrow$ ☺ change both s and l by 1. Such a decay would involve interaction with both the electric (to change l) and magnetic (to change s) components of the photon and is highly suppressed.

Problem 6 *Rohlf 18.11*

(a)

Consider this problem in the context of the theory of weak interactions as it existed before Glashow, Weinberg, Salam discovered how to unify it with the theory of electromagnetic interactions. In this context (Rohlf p. 509), the coupling constant for weak interactions is the Fermi constant G_F. According to Rohlf's Eq. (18.28), G_F has dimensions GeV fm^3. In a system of "natural" units in which $\hbar = c = 1$, we can transform a length (fm) into an inverse energy (inverse GeV) using the fact that $\hbar c \approx 0.2$ GeV fm. In natural units, G_F therefore has dimensions GeV^{-2}.

Since G_F is a coupling constant, like the fine structure constant α, it describes the strength of a quantum mechanical amplitude. The *rate* is proportional to the square of the modulus of this amplitude. Thus the decay rate W (which is inversely proportional to the muon lifetime τ_μ) is proportional to the square of G_F.

Now we have a dilemma. If $|G_F|^2$ were the only dimensionful component of W, W would have units of GeV^{-4}. However, using the fact that $\hbar = 6.6 \times 10^{-25}$ GeV sec, in natural units we know that W must have units of sec^{-1} or GeV. So far we are off by five powers of energy!

The solution is to bring in the only other relevant dimensionful quantity around, the muon mass m_μ. Remembering that mc^2 is the same as m in natural units, we find that we need five powers of m_μ in the numerator of W to make its units correct. Therefore its inverse, the muon lifetime, must have five powers of m_μ in its denominator.

Alternatively, we can approach this problem a bit more formally. Fermi's Golden rule says that the decay rate W is given by:

$$W = \frac{2\pi}{\hbar}|\mathcal{M}|^2 \times (\text{phase space})$$

where $\mathcal{M}$ is a transition amplitude obtained from perturbation theory (you'll learn all about this in 137B). In this case, all we need to know is that $\mathcal{M} \propto 1/M_W^2$. By dimensionality, we need to cancel the mass of the W boson with something, the best guess is the mass of the muon. The phase space available to the decay products in this case is proportional to the available energy, in other words the muon mass again. So

$$|\mathcal{M}|^2 \times (\text{phase space}) \propto \mathrm{m}_\mu^5$$

which again implies $\tau_\mu \propto m_\mu^{-5}$.

(b)

The tau particle has five times as many decay channels as the muon, so the phase space is increased by a factor of 5. So the tau lifetime is given, from the above arguments, by:

$$\tau_\tau = \tau_\mu \left(\frac{1}{5}\right)\left(\frac{105.7\ \text{MeV}}{1777\ \text{MeV}}\right)^5 = 0.3\ \text{ps}.$$

Problem 7 *Rohlf 19.18*

We have redshift parameter $z = 2$. We can employ the formula (19.15) on pg. 539 of Rohlf:

$$(1+z)^2 = \frac{1+\beta}{1-\beta}$$

where $\beta = v/c$ as usual. We can solve for β:

$$\beta = \frac{(1+z)^2 - 1}{(1+z)^2 + 1} = 0.8$$

and then use β in Hubble's law to determine the distance to the galaxy. Hubble's law is

$$d = \frac{\beta c}{H_0}$$

where H_0 is the Hubble constant. We find:

$$d = \frac{\beta c}{H_0} = \frac{0.8 \times (3 \times 10^8\ \text{m/s})}{7 \times 10^4\ m/s \cdot \text{Mpc}^-1} = 3400\ \text{Mpc}.$$

Problem 8 *Rohlf 19.29*

As a rough estimate, we simply set the thermal energy of particles in the early universe $\sim kT$ equal to the mass of 2 bottom quarks (actually a bottom and anti-bottom, which have the same mass). The bottom quarks must be produced in pairs so that "beauty" is conserved, since the b and $\bar{b}$ have equal and opposite beauties. You begin to wonder where this stuff comes from. Anyhow, the mass of 2 bottom quarks is 10 GeV, which implies $T = 10^{14}$ K. The characteristic expansion time t_{exp} comes from

$$t_{\text{exp}} = \frac{1}{H(t)} = \left(\frac{2.7\ \text{K}}{T}\right)^2 \sqrt{\frac{3c^2}{8\pi\rho G}} = 5 \times 10^{19}\ \text{s}\left(\frac{2.7\ \text{K}}{T}\right)^2.$$

You should probably check out the discussion on pp. 558-559 of Rohlf. So we can estimate:

$$t_{\text{exp}} \approx \frac{4\times 10^{20}\ \text{s}\cdot\text{K}^2}{T^2} = 4\times 10^{-8}\ \text{s}.$$

And that's all folks!
Good luck on your finals! Merry winter break!